Pure Math 30:

TRIGONOMETRY II

$$(\cos x - 1)(\tan x - 1) = 0$$

Lesson Ten

Solving Equations

Pure Math 30: EXPLAINED!

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Trigonometry Lesson 10

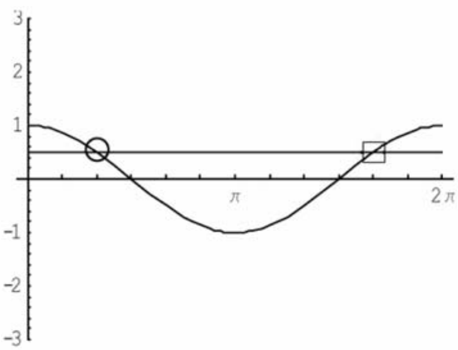
Part One - Graphically Solving Equations

Solving trigonometric equations graphically:

When a question asks you to solve a system of trigonometric equations, they are looking for the values of θ that make both equations true. There are two ways you can solve for θ: graphically in your TI-83, and algebraically. Part I will show the graphing method, and Parts II & III will focus on algebraic methods.

Example 1: Solve $\cos\theta = \frac{1}{2}$ and state the general solutions:

In your TI-83, graph each equation in degree mode.



Now use 2nd → Trace → Intersect to find the points of intersection. They occur at 60° & 300°

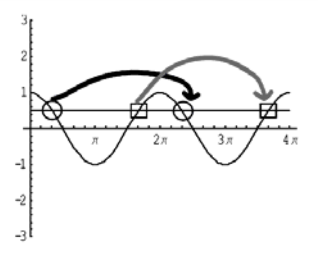
If you extend the window, you will see that the intersection points are in the same relative places, one period later.

The first general solution is:

$60° \pm n(360°)$ or $\frac{\pi}{3} \pm n(2\pi)$

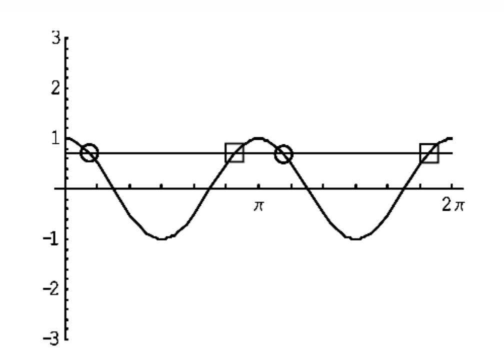
and the second is:

$300° \pm n(360°)$ or $\frac{5\pi}{3} \pm n(2\pi)$



Example 2: Solve $\cos 2\theta = \frac{\sqrt{2}}{2}$ and state the general solutions:

Graph both equations in your TI-83, then solve for the first two intersection points.



The first two intersection points are at 22.5° and 157.5°. As you can see in the graph, the solutions repeat themselves every period.

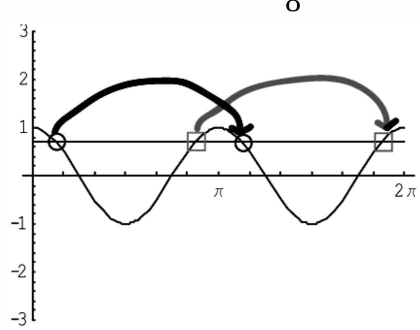
Since the b-value is 2, the period is 180°, or π.

The first general solution is:

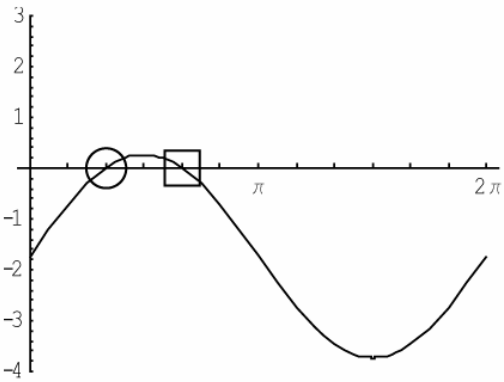
$22.5° \pm n(180°)$ or $\frac{\pi}{8} \pm n\pi$

And the second is:

$157.5° \pm n(180°)$ or $\frac{7\pi}{8} \pm n\pi$



Example 3: Graphically find the general solutions for $2\sin\theta - \sqrt{3} = 0$



Graph the two equations in your TI-83 and solve by finding the points of intersection.

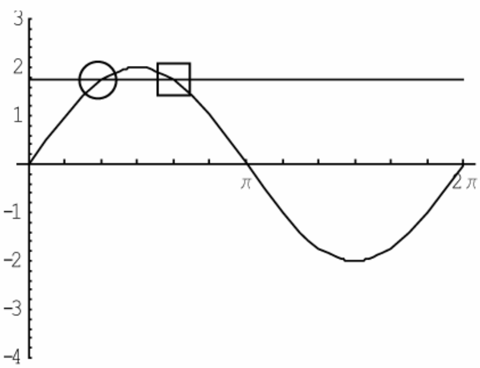
$60° \pm n(360°)$ or $\frac{\pi}{3} \pm n(2\pi)$

$120° \pm n(360°)$ or $\frac{2\pi}{3} \pm n(2\pi)$

(In this case, another method would be to find the x-intercepts using 2nd → Trace → Zero)

Note that even if you manipulate the equation, you can still solve by graphing:

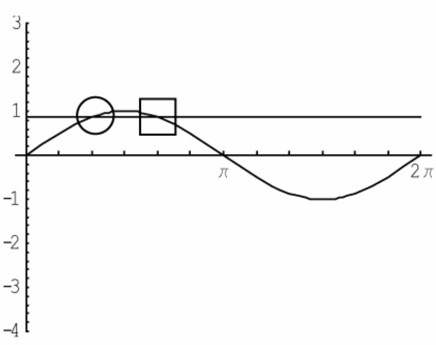
If you re-arrange the equation to $2\sin\theta = \sqrt{3}$ by taking $\sqrt{3}$ to the other side, we get:



Solving, we still have the same answers of 60° & 120°

If we manipulate the equation again by dividing both sides by 2, we get: $\sin\theta = \frac{\sqrt{3}}{2}$.

Solving this:



Once again, we still get the same answers of 60° & 120°

Manipulating an equation does NOT change the solution!

Find the general solution (In degrees & radian fractions) for each of the following equations:

1) $\sin 3x = \frac{\sqrt{3}}{2}$

2) $\cos^2 x = 1$

3) $\sin 2x = 0$

4) $\sin 4x = -\frac{1}{2}$

5) $\tan 2x = \sqrt{3}$

6) $\sin^2 x - 0.25 = 0$

7) $\tan x + \sqrt{3} = 0$

8) $\sin \frac{1}{2}x = -\frac{\sqrt{3}}{2}$

Trigonometry Lesson 10

Part One - Graphically Solving Equations

1)
$20^\circ \pm n(120^\circ)$ $\quad \frac{\pi}{9} \pm n\left(\frac{2\pi}{3}\right)$

$40^\circ \pm n(120^\circ)$ $\quad \frac{2\pi}{9} \pm n\left(\frac{2\pi}{3}\right)$

2) $\pm n(180^\circ)$ $\quad \pm \mathrm{n}\pi$

3) $\pm n(90^\circ)$ $\quad \pm \frac{\mathrm{n}\pi}{2}$

4)
$52.5^\circ \pm n(90^\circ)$ $\quad \frac{7\pi}{24} \pm \frac{n\pi}{2}$

$82.5 \pm n(90^\circ)$ $\quad \frac{11\pi}{24} \pm \frac{n\pi}{2}$

5) $30^\circ \pm n(90^\circ)$ $\quad \frac{\pi}{6} \pm n\left(\frac{\pi}{2}\right)$

6)
$30^\circ \pm n(180^\circ)$ $\quad \frac{\pi}{6} \pm n\pi$

$150^\circ \pm n(180^\circ)$ $\quad \frac{5\pi}{6} \pm n\pi$

7) $120^\circ \pm n(180^\circ)$ $\quad \frac{2\pi}{3} \pm n\pi$

8)
$480^\circ \pm n(720^\circ)$ $\quad \frac{8\pi}{3} \pm n(4\pi)$

$600^\circ \pm n(720^\circ)$ $\quad \frac{10\pi}{3} \pm n(4\pi)$

In this lesson, we will algebraically solve trigonometric equations. There are two main types of equations you will be asked to solve: linear & nonlinear.
Note that you will still be able to solve all of these in your calculator as you did in the previous lesson, but on the diploma they frequently have written response questions where you need to present an algebraic solution.

Linear Trigonometric Equations:

Example 1: Solve: $4\cos x + 2 = 0$ in the domain $0 \le x < 2\pi$

To solve this, we must get $\cos x$ by itself on the left side of the equation.

$$4\cos x + 2 = 0$$
$$4\cos x = -2$$
$$\cos x = \frac{-2}{4}$$
$$\cos x = -\frac{1}{2}$$

From the unit circle, we know $\cos x$ is $-\frac{1}{2}$ when $x = \frac{2\pi}{3}$ & $\frac{4\pi}{3}$

Example 2: Solve: $2\sin x\cos x = \sin x$ in the domain $0 \le x < 2\pi$

We need to bring everything over to the left side, then factor.

$$2\sin x\cos x - \sin x = 0$$
$$\sin x(2\cos x - 1) = 0$$

Now set each factor on the left side equal to zero:

$$\sin x = 0$$
$$x = 0, \pi$$

We don't include 2π since the domain is using a < sign, not a ≤ sign.

$$2\cos x - 1 = 0$$
$$2\cos x = 1$$
$$\cos x = \frac{1}{2}$$
$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

You may be tempted to divide both sides of the equation by $\sin x$ to cancel it out. Don't do this!
In math, you are never allowed to cancel variables on opposite sides of the equation.
Suppose you made this error and canceled $\sin x$. Then you would get:

$$2\sin x\cos x = \sin x$$
$$2\cos x = 1$$
$$\cos x = \frac{1}{2}$$
$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

As you can see, we lose solutions of 0 & π doing it this way.

Example 3: Solve: $\frac{\sin x}{4}+\frac{1}{12}=\frac{\sin x}{3}$ in the domain $0 \le x < 2\pi$

$$12\left(\frac{\sin x}{4}+\frac{1}{12}\right)=12\left(\frac{\sin x}{3}\right)$$
$$3\sin x+1=4\sin x$$
$$1=\sin x$$
$$x=\frac{\pi}{2}$$

Multiply both sides by the common denominator, which is 12. This will eliminate the fractions.

Example 4: Solve $\sin x\sec x\cot x=\sin x\sec x$ in the domain $0 \le x < 2\pi$

Bring all terms to one side so you can factor.

$$\sin x\sec x\cot x-\sin x\sec x=0$$
$$\sin x\sec x(\cot x-1)=0$$

$\sin x=0$
$x=0,\pi$

$\sec x=0$
$\frac{1}{\cos x}=0$
$x=$ No solutions

$\cot x-1=0$
$\cot x=1$
$x=\frac{\pi}{4},\frac{5\pi}{4}$

The combined solution set is:
$0,\frac{\pi}{4},\pi,\frac{5\pi}{4}$

Example 5: Solve $3\sin x=2$ in the domain $0 \le x < 2\pi$

$3\sin x=2$
$\sin x=\frac{2}{3}$

Since $\frac{2}{3}$ is not on the unit circle, we are forced to do this equation graphically in the calculator.

41.8° = 0.73 rad
138.2° = 2.41 rad

Solve each of the following in the domain $0 \le x < 2\pi$

1) $\sin x - \frac{1}{2} = 0$

2) $2\cos x - \sqrt{3} = 0$

3) $4\sin x + 3 = 3\sin x + 2$

4) $2\sin x\cos x = \cos x$

5) $3\tan x = 3$

6) $2\cos x + 1 = 2$

7) $\sin x\cos x\tan x + \sin x\cos x = 0$

8) $2\cos x(\cos x + \frac{1}{2}) = 0$

9) $(\sec x+1)(\sin x-1)(\cot x-1)=0$

10) $\csc x\cot x+\cot x=0$

11) $\tan x\cos x+\cos x=0$

12) $(\cos x-1)(\tan x-1)=0$

13) $\sec x\cos x+\sec x=0$

14) $\frac{\sin x}{2}=\frac{\sin x}{3}$

15) $\frac{\tan x}{2}-\frac{\tan x}{3}=\frac{-1}{6}$

16) $\frac{\csc x}{5}+\frac{\csc x}{3}=\frac{16}{15}$

1) $x = \frac{\pi}{6}, \frac{5\pi}{6}$

2) $x = \frac{\pi}{6}, \frac{11\pi}{6}$

3) $x = \frac{3\pi}{2}$

4)
$$2\sin x\cos x - \cos x = 0$$
$$\cos x(2\sin x - 1) = 0$$
$$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$$

5)
$$\tan x = 1$$
$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

6)
$$\cos x = \frac{1}{2}$$
$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

7)
$$\sin x\cos x(\tan x + 1) = 0$$
$$x = 0, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{3\pi}{2}, \frac{7\pi}{4}$$

8) $x = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}$

9) $x = \frac{\pi}{4}, \frac{\pi}{2}, \pi, \frac{5\pi}{4}$

10)
$$\cot x(\csc x + 1) = 0$$
$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

11)
$$\cos x(\tan x + 1) = 0$$
$$x = \frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$$

12) $x = 0, \frac{\pi}{4}, \frac{5\pi}{4}$

13)
$$\sec x(\cos x + 1) = 0$$
$$x = \pi$$

14)
$$3\sin x = 2\sin x$$
$$3\sin x - 2\sin x = 0$$
$$\sin x = 0$$
$$x = 0, \pi$$

15)
$$6\left(\frac{\tan x}{2} - \frac{\tan x}{3}\right) = 6\left(\frac{-1}{6}\right)$$
$$3\tan x - 2\tan x = -1$$
$$\tan x = -1$$
$$x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

16)
$$15\left(\frac{\csc x}{5} + \frac{\csc x}{3}\right) = 15\left(\frac{16}{15}\right)$$
$$3\csc x + 5\csc x = 16$$
$$8\csc x = 16$$
$$\csc x = 2$$
$$\sin x = \frac{1}{2}$$
$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

Quadratic Trigonometric Equations:

Example 1: Solve $4\sin^2 x - 3 = 0$ in the domain $0 \le x < 2\pi$

$4\sin^2 x = 3$

$\sin^2 x = \frac{3}{4}$

$\sqrt{\sin^2 x} = \sqrt{\frac{3}{4}}$

$\sin x = \pm\frac{\sqrt{3}}{2}$

$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

Example 2: Solve $\cos^2 x - \cos x = 0$ in the domain $0 \le x < 2\pi$

$\cos^2 x - \cos x = 0$

$\cos x(\cos x - 1) = 0$

$\cos x = 0$	$\cos x - 1 = 0$
$x = \frac{\pi}{2}, \frac{3\pi}{2}$	$\cos x = 1$
	$x = 0$

The complete solution is: $x = 0, \frac{\pi}{2}, \frac{3\pi}{2}$

Example 3: Solve $2\cos^2 x = 3\cos x - 1$ in the domain $0 \le x < 2\pi$

$2\cos^2 x - 3\cos x + 1 = 0$

$(2\cos x - 1)(\cos x - 1) = 0$

$2\cos x - 1 = 0$	
$2\cos x = 1$	$\cos x - 1 = 0$
$\cos x = \frac{1}{2}$	$\cos x = 1$
$x = \frac{\pi}{3}, \frac{5\pi}{3}$	$x = 0$

The complete solution is: $x = 0, \frac{\pi}{3}, \frac{5\pi}{3}$

Example 4: Solve $\sin^3 x - 5\sin^2 x + 6\sin x = 0$ in the domain $0 \le x < 2\pi$

$\sin x(\sin^2 x - 5\sin x + 6) = 0$

$\sin x(\sin x - 2)(\sin x - 3) = 0$

$\sin x = 0$	$\sin x - 2 = 0$	$\sin x - 3 = 0$
$x = \frac{\pi}{2}, \frac{3\pi}{2}$	$\sin x = 2$	$\sin x = 3$
	$x =$ no solution	$x =$ no solution

Example 5: Solve $\tan^8 x - \tan^4 x = 0$ in the domain $0 \le x < 2\pi$, and state the general solution

Watch out for difference of squares!

$\tan^4 x(\tan^4 x - 1) = 0$

$\tan^4 x(\tan^2 x + 1)(\tan^2 x - 1) = 0$

$\tan^4 x(\tan^2 x + 1)(\tan x + 1)(\tan x - 1) = 0$

$\tan^4 x = 0$	$\tan^2 x + 1 = 0$	$\tan x + 1 = 0$	$\tan x - 1 = 0$
$\sqrt[4]{\tan^4 x} = \sqrt[4]{0}$	$\tan^2 x = -1$	$\tan x = -1$	$\tan x = 1$
$\tan x = 0$	No solution	$x = \frac{3\pi}{4}, \frac{7\pi}{4}$	$x = \frac{\pi}{4}, \frac{5\pi}{4}$
$x = 0, \pi$			

The complete solution set for $0 \le x < 2\pi$ is: $x = 0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}$

The general solution is: $x = \pm n\pi$ and $x = \frac{\pi}{4} \pm n\frac{\pi}{2}$

These two general solutions will account for all the angles we found.

Example 6: Solve $3\cot^2 x - \cot x - 1 = 0$ in the domain $0 \le x < 2\pi$

This equation cannot be factored, so graph and find the solutions in radian decimals:

0.92 rad, 1.98 rad, 4.06 rad, 5.12 rad

Graph in your TI-83 as:

$$\frac{3}{(\tan(x))^2} - \frac{1}{\tan(x)} - 1$$

Then use 2nd → Trace → Zero to find x-intercepts.

Solve each of the following in the domain $0 \le x < 2\pi$

1) $\cos^2 x = \frac{3}{4}$

2) $\sin^2 x - \frac{1}{4} = 0$

3) $3\tan^2 x = 3$

4) $4\sin^2 x - 3 = 0$

5) $2\sin^2 x - \sin x - 1 = 0$

6) $2\sin^2 x = 1 - \sin x$

7) $6\cos^2 x - 3\cos x - 3 = 0$

8) $2\sin^2 x - 3\sin x + 1 = 0$

9) $4\cos^2 x + 2\cos x - 2 = 0$

10) $2\cos^3 x + \cos^2 x - \cos x = 0$

11) $\tan^4 x - \tan^2 x = 0$

12) $\cos^8 x - \cos^4 x = 0$

ANSWERS:

1) $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

2) $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

3) $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

4) $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

5)
$(2\sin x + 1)(\sin x - 1) = 0$
$x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

6)
$(2\sin x - 1)(\sin x + 1) = 0$
$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

7)
$3(2\cos^2 x - \cos x - 1) = 0$
$3(2\cos x + 1)(\cos x - 1) = 0$
$x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$

8)
$(2\sin x - 1)(\sin x - 1) = 0$
$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

9)
$2(2\cos^2 x + \cos x - 1) = 0$
$2(2\cos x - 1)(\cos x + 1) = 0$
$x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

10)
$\cos x(2\cos^2 x + \cos x - 1) = 0$
$\cos x(2\cos x - 1)(\cos x + 1) = 0$
$x = \frac{\pi}{3}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \frac{5\pi}{3}$

11)
$\tan^2 x(\tan^2 x - 1) = 0$
$\tan^2 x(\tan x + 1)(\tan x - 1) = 0$
$x = 0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}$

12)
$\cos^4 x(\cos^4 x - 1) = 0$
$\cos^4 x(\cos^2 x + 1)(\cos^2 x - 1) = 0$
$\cos^4 x(\cos^2 x + 1)(\cos x + 1)(\cos x - 1) = 0$
$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

Part Four - Algebraically Solving Multiple Angles

Algebraically Solving Double, Triple, and Half Angles:

While technology can be used to solve equations involving double & triple angles, it is advantageous to understand the algebraic process involved with these question types.

Example 1: Solve $\sin 2\theta = \frac{\sqrt{3}}{2}$ **algebraically over the interval $0 \le x \le 2\pi$.**

To solve this equation algebraically, you need to perform the following steps:

Step 1) Start by solving the equation $\sin x = \frac{\sqrt{3}}{2}$. We can do this easily using the unit circle. The answer to this equation is $x = \frac{\pi}{3}$ and $\frac{2\pi}{3}$.

Step 2) Add 2π to each of the angles we found.

$$\frac{\pi}{3} + 2\pi = \frac{\pi}{3} + \frac{6\pi}{3} = \frac{7\pi}{3}$$
$$\frac{2\pi}{3} + 2\pi = \frac{2\pi}{3} + \frac{6\pi}{3} = \frac{8\pi}{3}$$

Don't forget that adding fractions requires a common denominator.

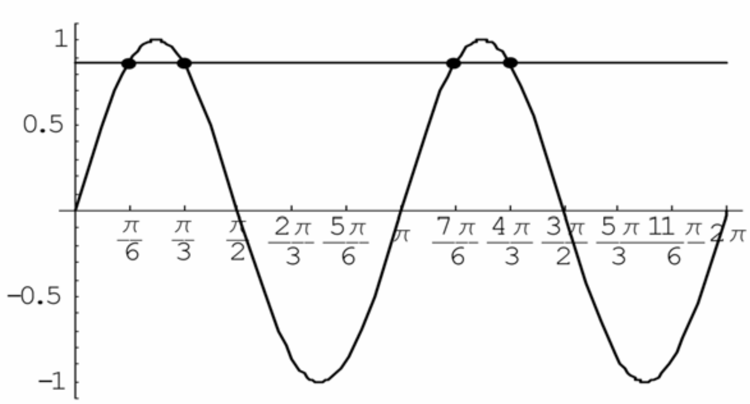
Step 3) Finally, take all your solutions $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$ and divide by 2. *(Or multiply by ½)*

The answer is $\mathbf{\frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}}$

The general solution is:

$$x = \begin{Bmatrix} \frac{\pi}{6} + k\pi, k \in I \\ \frac{\pi}{3} + k\pi, k \in I \end{Bmatrix}$$

The period of $\sin 2\theta$ (*which is* π) goes here.



You can verify the results by graphing and checking the x-coordinates of the points of intersection.

<u>Example 2:</u> Solve $\cos 3\theta = \frac{\sqrt{2}}{2}$ for the domain $0 \le \theta \le 2\pi$

Step 1) Start by solving the equation $\cos x = \frac{\sqrt{2}}{2}$. We can do this using the unit circle.

The answer to this equation is $x = \frac{\pi}{4}$ and $\frac{7\pi}{4}$.

Step 2) Add 2π to each of the angles we found.

$$\frac{\pi}{4} + 2\pi = \frac{\pi}{4} + \frac{8\pi}{4} = \frac{9\pi}{4}$$
$$\frac{7\pi}{4} + 2\pi = \frac{7\pi}{4} + \frac{8\pi}{4} = \frac{15\pi}{4}$$

Step 3) Add 2π to each of the angles we found in Step 2.

$$\frac{9\pi}{4} + 2\pi = \frac{9\pi}{4} + \frac{8\pi}{4} = \frac{17\pi}{4}$$
$$\frac{15\pi}{4} + 2\pi = \frac{15\pi}{4} + \frac{8\pi}{4} = \frac{23\pi}{4}$$

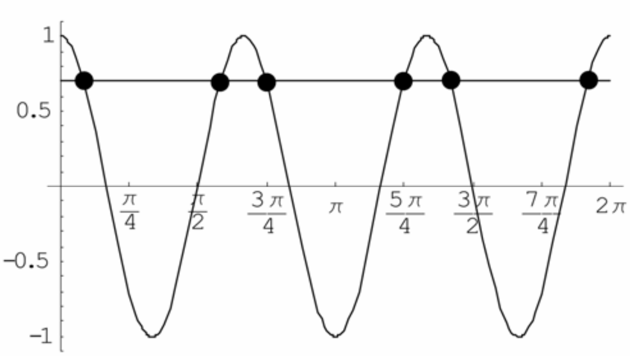
Step 4) Finally, take all your solutions $\frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \frac{17\pi}{4}, \frac{23\pi}{4}$ and divide by 3. *(Or multiply by 1/3)*

The answer is $\boldsymbol{\frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{23\pi}{12}}$

The general solution is:

$$x = \begin{Bmatrix} \frac{\pi}{12} + k\frac{2\pi}{3}, k \in I \\ \frac{7\pi}{12} + k\frac{2\pi}{3}, k \in I \end{Bmatrix}$$

The period of cos3θ (*which is 2π/3*) goes here.



You can verify the results by graphing and checking the x-coordinates of the points of intersection.

Part Four - Algebraically Solving Multiple Angles

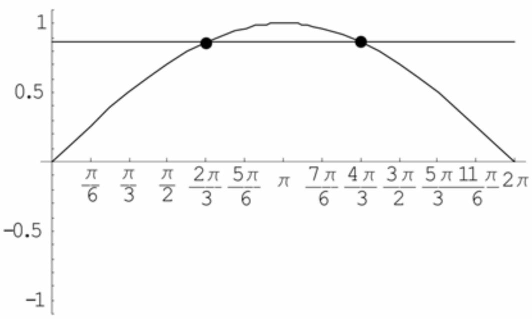
Example 3: Solve $\sin\frac{1}{2}\theta = \frac{\sqrt{3}}{2}$ for the domain $0 \le \theta \le 2\pi$

To solve this equation algebraically, you need to perform the following steps:

Step 1) Start by solving the equation $\sin x = \frac{\sqrt{3}}{2}$. Do this using the unit circle.

The answer to this equation is $x = \frac{\pi}{3}$ *and* $\frac{2\pi}{3}$.

Step 2) Divide each angle by $\frac{1}{2}$ *(Or multiply by 2)* to obtain $\frac{2\pi}{3}$ *and* $\frac{4\pi}{3}$.



You can verify the results by graphing and checking the x-coordinates of the points of intersection

The general solution is:

$$x = \begin{Bmatrix} \frac{2\pi}{3} + k(4\pi), k \in I \\ \frac{4\pi}{3} + k(4\pi), k \in I \end{Bmatrix}$$

The period of $\sin(0.5\theta)$ (*which is* 4π) goes here.

Questions: Algebraically solve for θ over the interval $0 \le \theta \le 2\pi$

1) $\sin 2\theta = -\frac{\sqrt{3}}{2}$

2) $\cos 2\theta = \frac{\sqrt{3}}{2}$

3) $\cos 2\theta = -\frac{1}{2}$

4) $\sin 2\theta = -1$

5) $\cos 3\theta = \frac{\sqrt{3}}{2}$

6) $\cos\left(\frac{1}{2}\theta\right) = \frac{\sqrt{2}}{2}$

Answers:

1) $\theta = \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6}$

2) $\theta = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$

3) $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

4) $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

5) $\theta = \frac{\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{23\pi}{18}, \frac{25\pi}{18}, \frac{35\pi}{18}$

6) $\theta = \frac{\pi}{2}$