6.8 Linear Combination & Spanning Sets

(A) <u>Definition</u> : <u>Collinear Vectors</u>

- when two vectors are parallel & lie on the same line
- two vectors are scalar multiples of each other
- can be written as $\vec{u} = k\vec{v}$, where $\vec{u}$ is a scalar multiple of $\vec{v}$ & $k \in \mathbb{R}$

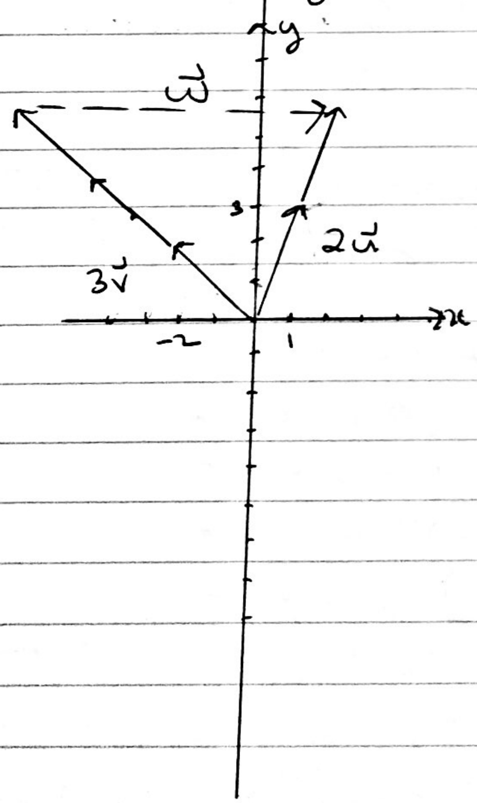
(B) <u>Definition</u>: <u>Linear Combination in $\mathbb{R}^2$</u> (2-Dimension)

- A linear combination of two vectors $\vec{u}$ & $\vec{v}$ can be written as

$$\vec{w} = a\vec{u} + b\vec{v}, \quad \text{where } a \text{ \& } b \text{ are scalars \& } \vec{u} \text{ \& } \vec{v} \text{ are non-colinear vectors}$$

ie: $\vec{w} = 2\vec{u} - 3\vec{v}$, where $\vec{u} = [1,3]$, $\vec{v} = [-2,2]$

<u>Graphically</u>



<u>Algebraically</u>

$$\vec{w} = 2[1,3] - 3[-2,2]$$
$$= [2+6,\ 6-6]$$
$$= [8,0]$$

or $\vec{w}$ can be written as

$$\vec{w} = 8\hat{i}, \text{ where } \hat{i} = [1,0]$$

<u>Note</u>: Any vector can be written in terms of it's unit vectors. We say $\hat{i}$ & $\hat{j}$ SPAN $\mathbb{R}^2$ or $\{(\hat{i}, \hat{j})\}$ forms a spanning set in $\mathbb{R}^2$.

Question 1: Show that $\vec{w} = [10,12]$ is a linear combination of $\vec{u} = [2,-4]$ & $\vec{v} = [1,-6]$.

$$\Rightarrow \quad \vec{w} = a\vec{u} + b\vec{v}$$
$$[10,12] = a[2,-4] + b[1,-6]$$

so that

$$\left.\begin{aligned} 10 &= 2a + 1b \\ \& \quad 12 &= -4a - 6b \end{aligned}\right\} \begin{matrix} (1) \\ (2) \end{matrix}$$

produces a linear system thus use elimination/substitution to find a & b.

$$\left.\begin{aligned} 20 &= 4a + 2b \\ 12 &= -4a - 6b \end{aligned}\right\} \begin{matrix} (3) \\ (2) \end{matrix}$$

$$32 = -4b \qquad (3) + (2)$$
$$-8 = b$$

use (1) sub. $b = -8$: $10 = 2a - 8 \Rightarrow a = 9$

$\therefore$

$$[10,12] = 9[2,-4] + (-8)[1,-6]$$

Question 2: Do the set of vectors $\{[3,9]$ & $[9,27]\}$ span $\mathbb{R}^2$?

This means can we find a vector $\vec{w}$ that can be written as a linear combination of $[3,9]$ & $[9,27]$.

$$\Rightarrow \quad \vec{w} = a[3,9] + b[9,27]$$
$$= [3a + 9b, 9a + 27b]$$
$$= [3(a+3b), 9(a+3b)]$$
$$= [3,9](a+3b)$$
$$= (a+3b)[3,9]$$

This shows that $[3,9]$ & $[9,27]$ can only be written as scalar multiples of $[3,9]$ thus they are parallel and non-collineare ; $\therefore$ Can not span $\mathbb{R}^2$.

Question 3: Show that $\{[3,2]$ & $[-1,3]\}$ is a spanning set in $\mathbb{R}^2$.

$\Rightarrow$ Let $\vec{w} = [x,y]$ so that $[x,y] = a[3,2] + b[-1,3]$

$x = 3a - b$ — ① $\Rightarrow$ $b = 3a - x$ — ③ sub. into ②
$y = 2a + 3b$ — ②

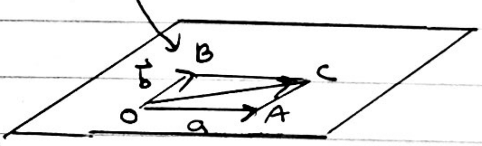
$y = 2a + 3(3a - x)$ sub. into ②
$y = 11a - 3x$ ~~$y = 2\left(\frac{y+3x}{11}\right) + 3(3a-x)$~~
$\frac{y + 3x}{11} = a$

sub into ③ $b = 3a - x$
$= 3\left(\frac{y+3x}{11}\right) - x$
$= \frac{3y + 9x - 11x}{11}$
$b = \frac{3y - 2x}{11}$

$\therefore$ if given any (x,y) the values of a, b will be unique $\therefore$ $[3,2]$ & $[-1,3]$ span $\mathbb{R}^2$

(G) Linear Combination in $\mathbb{R}^3$

Any pair of non-zero, non-collinear vectors will span a plane in $\mathbb{R}^3$



$\vec{OC} = \vec{OA} + \vec{OB}$
$\vec{w} = m\vec{a} + n\vec{b}$

We say $\vec{OC}$ lies on the plane as so does $\vec{OA}$ & $\vec{OB}$
Vectors $\vec{OA}, \vec{OB}, \vec{OC}$ are said to be COPLANAR (all lie in the same plane).

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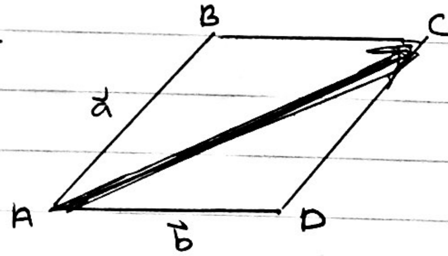
$\vec{AB} + \vec{AD} = \vec{AC}$
$m\vec{a} + n\vec{b} = \vec{AC}$



A(-1,3,4)
B(-2,3,-1) $\quad \vec{AB} = [-1,0,-5] = \vec{a}$
C(-5,6,x) $\quad \vec{AD} = [1,-3,-4] = \vec{b}$
D(0,0,0) $\quad \vec{AC} = [-4,3,x-4]$

$$m\vec{a} + n\vec{b} = \vec{AC}$$

$$m[-1,0,-5] + n[1,-3,-4] = [-4,3,x-4]$$

$$\left.\begin{aligned} -1m + n &= -4 \quad ① \\ -3n &= 3 \quad ② \\ -5m - 4n &= x-4 \quad ③ \end{aligned}\right\} \Rightarrow n = -1$$

so $-1+4 = m$
$3 = m$

sub into ③ $\quad -5(3) - 4(-1) = x-4$
$-15+4+4 = x$
$\underline{\underline{-7 = x}}$

$\therefore$ A, B, C, D all lie in the same plane.

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$$\vec{a} = m\vec{b} + n\vec{c}$$

$$[1,12,-29] = m[3,1,4] + n[1,2,-3]$$

$$\left.\begin{aligned} 1 &= 3m + n \quad ① \times -2 \\ 12 &= m + 2n \quad ② \\ -29 &= 4m - 3n \quad ③ \end{aligned}\right\}$$

$\Rightarrow$
$-2 = -6m + (-2n)$ — ④
$12 = m + 2n$ — ②
$10 = -5m$ $\quad$ ④+②
$m = -2$

sub. $m=-2$ into ①
$1 = 3(-2) + n$
$7 = n$

check $\bar{w}$ ③ $\quad 4(-2) - 3(7) = -29$ ✓

$\therefore \vec{a} = m\vec{b} + n\vec{c}$ is a linear combination